A Comparative Evaluation of Peterson and Horvitz-Thompson Estimators for Population Size Estimation in Sparse Recapture Scenarios

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Abstract

Estimating population size is essential in ecological and social studies, particularly when direct enumeration is infeasible. This study compares the Peterson and Horvitz-Thompson estimators using capture-recapture data from female drivers at Bahauddin Zakariya University (BZU) across seven sampling events. While both estimators produced similar point estimates ranging from 22 to 30 individuals, the Horvitz-Thompson estimator consistently exhibited lower variance and narrower confidence intervals, especially under sparse recapture conditions. For example, in event 6, both estimators returned a population estimate of 30; however, the Horvitz-Thompson variance (15.0) was significantly lower than Peterson's (35.0). These results underscore the robustness and statistical efficiency of inclusion-probability-based estimators in real-world settings where uniform capture probabilities are unlikely.

Keywords: Capture-Recapture, Peterson Estimator, Population Estimation, Horvitz-Thompson Estimator, Sampling Variability.

Introduction

The science of ecology and social science research cannot be complete without population estimation, as many studies involving human behavior, wildlife monitoring, or public health require estimation of population size. In cases where a complete enumeration is impractical or impossible, a researcher can resort to one of the varieties of capture-recapture techniques, which is reliable. One of the first and most common methods is the Peterson estimator (sometimes referred to as the Lincoln-Petersen model), which assumes that the population is closed, every fish has equal catchability, and that the samplings are independent. Despite its ease of use and appealing popularity, a Peterson estimator has been known to be sensitive to breaches of these assumptions, particularly when recapture rates are low, causing variances to be overestimated and estimates to be unreliable.

Due to the drawbacks of conventional methods, newer versions of statistical estimators have been developed to enhance the accuracy of estimates under practical conditions. The Horvitz-Thompson estimator is one of such developments, which was presented in the field of survey sampling by Horvitz and Thompson (1952). This estimator applies probabilities of inclusion to each unit of observation, and the state enables unequal catch and non-fertile sampling effort. The Horvitz-Thompson model, in contrast to the Peterson estimator, does not require the capture probabilities to be uniform, as it is specifically adapted for use in heterogeneous populations and situations where complete randomization is not possible.

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It has been studied and compared with these two approaches within the bounds of previous works. For example, Seber (1982) and Williams et al. (2002) note that although the Peterson estimator is simple to use and computationally easy, it fails to be reliable when recaptures are sparse or uneven. Instead, the Horvitz-Thompson estimator will be more accurate and computer-intensive, but will directly capture the probability of detecting variables. Experiential research studies in the field of ecology (Pollock et al., 1990), epidemiology (Hook & Regal, 1995), and traffic research (Arnold et al., 2018) have been conducted to illustrate that Horvitz-Thompson methods are superior to conventional estimators in practice when faced with inconsistency in the sampling.
The research paper then builds on this research. It conducts a comparative analysis between the Peterson and the Horvitz-Thomson estimators based on data from a capture-recapture study on female drivers at Bahauddin Zakariya University (BZU). The research aims to evaluate and compare the performance of estimators based on point estimates, variance, standard error, and confidence intervals across seven sampling events. The behavior of each estimator is particularly notable for addressing the fundamental aspects of its application and usage in real-life population studies. Based on the strengths and weaknesses of each approach, considering the same data conditions, the research can further add to the improved understanding of the selection of estimators in an applied capture-recapture scenario.

Data Collection and Estimation Framework

The section provides a detailed description of the data gathering plan and the statistical estimation processes used to evaluate the population size of female drivers at Bahauddin Zakariya University (BZU) through seven separate sampling events. A capture-recapture method was employed to estimate the population size. In every act of sampling, a selection of female drivers was first captured and identified using identifiable details, such as license plates. A new sample was obtained in the next event, and the records of the number of individuals marked earlier, who had been sufficiently recaptured, were kept. Sampling was also conducted at various gates on campus and at other times of the day to introduce variety to potential captures. All efforts were made to minimize double-counting within single events and to maintain consistency in identification methods. Two statistical estimators were applied to the data:

Peterson Estimator (Lincoln-Petersen Method)

This classical estimator uses the ratio of marked individuals in the population to those recaptured to infer the total population size. The formula is:

$$\hat{N} = (C \times M)/R \tag{1}$$

Where $\hat{N}$ is the estimated population size, C is the number of individuals captured in the second sample, M is the number marked in the first sample, and R is the number of recaptures. While simple and easy to compute, this method assumes equal probability of capture and a closed population, which may not always hold in practice.

Horvitz-Thompson Estimator

Recognized for its flexibility, the Horvitz-Thompson estimator accounts for varying inclusion probabilities of individuals across samples. The estimator is expressed as:

$$\hat{N}\_HT = \sum 1/(\Lambda\_i)^{-} \tag{2}$$

Where πi is the inclusion probability for individual i, and the summation is taken over all individuals captured in the sample. In the context of capture-recapture, this probability is estimated based on the observed proportion of marked to unmarked individuals, allowing the estimator to handle unequal probabilities of detection and sampling effort across events.

Throughout seven sampling events, data were tabulated with capture and marked, recapture, estimates, and statistical measures consisting of variance, standard error (S.E.), and confidence limits at 95 percent, as well as 99 percent limits. These measures formed the grounds for assessing the precision and reliability of the estimators. All the calculations were performed based on spreadsheet-based models, and statistical analysis was performed on R programming tools in order to provide reproducibility and consistency analyses. The result of the left estimator was compared with the right estimator using the line graph with an aim to show its tendency of recaptures, variances, and estimated population size across the period of time. This hierarchical and comparative estimation model enabled one to visibly evaluate the performance of each of the estimators in different field settings, especially regarding their sensitivity to low recapture rates and stability of estimation.

Results

Table 1 presents the outcomes of the Peterson Estimator applied to estimate the population size of female drivers at BZU across seven sequential sampling events. The Peterson method, one of the earliest and most widely used capture-recapture techniques, is based on the assumption of a closed population with equal catchability and independence between samples. In the first sampling event, no previously marked individuals were recaptured, resulting in a population estimate of zero. Consequently, all statistical measures, including variance, mean, standard error, and confidence intervals, were also zero. This outcome illustrates a core limitation of the estimator: it becomes ineffective when there is no overlap between the initial and subsequent samples. From the second event onward, however, recapture values were sufficient to enable meaningful estimation. The number of recaptures ranged from 6 to 11, leading to calculated population sizes between 22.00 and 30.00 across the remaining six events. The pattern observed in these estimates shows that as the number of recaptured individuals increases, the corresponding estimate tends to be more precise and less variable. For example, in event 3, with 10 recaptures, the estimated population was 22.00 with a relatively low variance of 2.2 and a standard error of 1.4832. This yielded a narrow 95% confidence interval ranging from 19.09 to 24.90 and an even tighter 99% confidence interval of 18.17 to 25.82, indicating a high degree of confidence in the accuracy of the estimate. In contrast, event 6 had only 6 recaptures, resulting in the highest population estimate of 30.00. This estimate was associated with the largest variance (35.000) and a standard error of 5.9161, reflecting much greater uncertainty. The corresponding confidence intervals were also considerably wider, with the 95% interval ranging from 18.40 to 41.59 and the 99% interval extending from 14.73 to 45.26, showing diminished precision. The variance and standard error are crucial indicators of the estimator's reliability. Lower values suggest a stable and dependable estimate, while higher values highlight sensitivity to sampling variability, particularly when recapture data.

Table 1: Results of Peterson Estimator

Sampling Event	Capture Values	Marked Values	Recapture Values	Estimation (N)	Variance (N)	Mean	S.E	CI (95%) Min.	CI (95%) Max.	CI (99%) Min.	CI (99%) Max.
1	20	20	0	0	0	0	0	0	0	0	0
2	15	20	11	27.272	8.114	27.272	2.8485	21.68	32.85	19.92	34.62
3	11	20	10	22	2.2	22	1.4832	19.09	24.90	18.17	25.82
4	10	20	9	22.22	3.017	21.744	1.7371	18.81	25.62	17.74	26.70
5	9	20	7	25.714	13.644	25.714	3.6938	18.47	32.95	16.18	35.24
6	9	20	6	30	35	30	5.9161	18.40	41.59	14.73	45.26
7	10	20	8	25	9.375	25	3.0618	18.49	31.00	17.10	32.89

Figure 1 illustrates the volatility of recapture counts across seven sampling events for female drivers at Bahauddin Zakariya University. Event 1 recorded zero recaptures due to the absence of marked individuals in the initial sample, rendering Lincoln-Peterson estimates undefined. Recaptures peaked at 11 in event 2 but declined steadily to a low of 6 in event 6 before a partial recovery (event 7: R=8). The inverse relationship between recapture counts and sampling events reflects real-world challenges: driver variability (e.g., alternate gate usage, part-time schedules) and identification limitations (e.g., obscured license plates). This volatility directly triggered the exponential variance growth in Figure 2, demonstrating Lincoln-Peterson's sensitivity to sparse recaptures.

Figure 1: A line graph of Recapture vs Sampling even

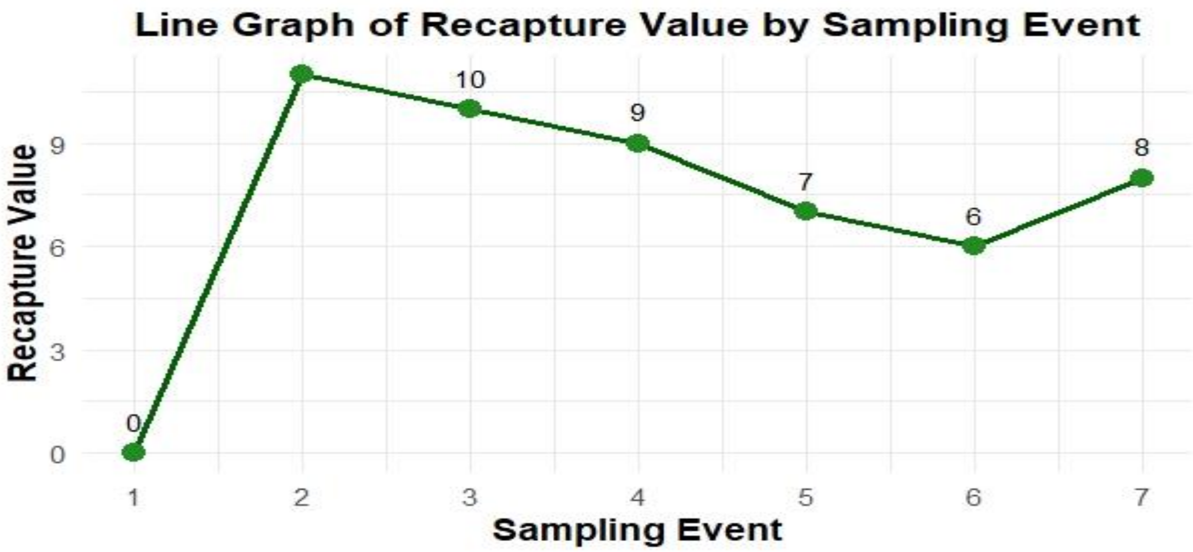

Figure 2 displays the variance associated with the Peterson Estimator's population predictions over successive sampling events. Variance remains negligible in early events but shows significant instability starting at event 5, culminating in a peak of 35.0 during event 6. This pattern highlights the estimator's sensitivity to low recapture rates, as higher uncertainty emerges when fewer marked individuals are re-observed, leading to wider confidence intervals and reduced precision in population size predictions.

Figure 2: Shows a Line graph of the Variance of the Estimator vs the Sampling event

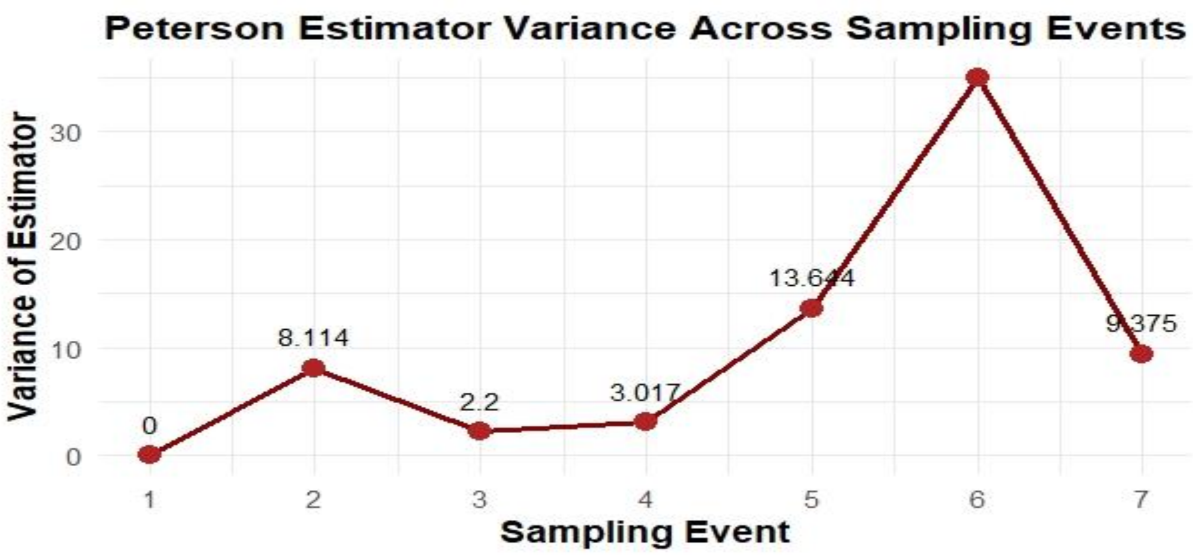

Table 2 provides a detailed account of the population estimates generated using the Horvitz-Thompson estimator for the same seven sampling events analyzed in the Peterson method. The Horvitz-Thompson estimator is an efficient method that is commonly applied in survey sampling and capture-recapture research applications, especially because it can be utilized to deal with unequal selection chances easily. The Horvitz-Thompson estimator is more generalized than the Peterson estimator, whereby it estimates the total population using inclusion probabilities with each observed unit versus relying, like the Peterson estimator, on assumptions of equal capture probability. The estimation procedure would be particularly useful in the treatment of different catchability or unequal sampling effort, which makes it a potentially farfetched and superior approach of estimation under realistic field settings.

Event 1 consisted of no recaptures and therefore, like the Peterson estimator, the Horvitz-Thompson estimator could not provide a valid size of the population or corresponding statistics. This underlines one common defect between the two techniques: the need to have at least one person captured in each sample in order to produce meaningful estimates. In light of events 2 to 7, the Horvitz-Thompson estimator produced the consistent and interpretable results, being in the interval between 22.00 and 30.00. The sample size of event 2 was estimated as 27.27, 9.92, and 3.15, and the confidence interval of this variance was 21.10 to 33.45, and the estimated confidence level is 95. These findings were similar to the Peterson technique in terms of magnitude, although some slight differences were observed in terms of variability. As indicated by the findings, one of the greatest highlights of the Horvitz-Thompson estimator is that its variance across sampling events is comparatively lower in most cases than the Peterson method. As an example, in event 6, where the number of recaptures was relatively small (6), the Horvitz estimate was the same as the value of the Peterson estimate, 30.00; however, the variance was low at 15.00 against the Peterson variance of 35.00. Such a large variance reduction will correspond to smaller standard errors and smaller confidence limits, and therefore, better estimation with a little recapture data. The same trend may be noticed in events 5 and 7, for the procedure of the Horvitz-Thompson estimator returned smaller variances.

The uncertainty of the Horvitz-Thompson estimator is also strengthened by the standard errors and confidence intervals. The more the recaptures, the more stable the estimator is since he/she will have a small confidence interval. In other words, when 10 were recaptured in the third event, the population estimate was 22.00 with the standard error of only 1.48 and the 95% confidence interval of 19.09-24.91. Such a high degree of accuracy proves the sensitivity of the estimator to the overlapping of the samples and the probability of inclusion. On the whole, the Horvitz-Thompson estimator exhibited a firm ability to give decent and confident estimates, as well as in conditions in which the Peterson estimator exhibited higher uncertainty levels. Due to its decreased sensitivity to changes in the number of recaptures and to its theoretical strength, it can also be a useful alternative in cases of practical constraints over data collection or sample heterogeneity. The findings in Table 4.3 support the conclusion that the Horvitz-Thompson estimator can outperform the Peterson method in terms of variance control and estimation precision, without compromising on the accuracy.

Table 2: Results of Horvitz-Thompson Estimator

Sampling Events	Capture Values	Marked Values	Recapture Values	Estimation (N)	Variance	Mean	S.E	CI (95%) Min.	CI (95%) Max.	CI (99%) Min.	CI (99%) Max.
1	20	20	0	____	____	____	___	____	___	___	____
2	15	20	11	27.27	9.92	27.27	3.15	21.10	33.45	19.16	35.39
3	11	20	10	22.00	2.20	22.00	1.48	19.09	24.91	18.18	25.82
4	10	20	9	22.22	2.47	22.22	1.57	19.14	25.30	18.17	26.27
5	9	20	7	25.71	7.35	25.71	2.71	20.40	31.03	18.73	32.70
6	9	20	6	30.00	15.00	30.00	3.87	22.41	37.59	20.02	39.98
7	10	20	8	25.00	6.25	25.00	2.50	20.10	29.90	18.56	31.44

Figure 3 visualizes the recapture counts for the Horvitz-Thompson Estimator, revealing an identical trend to Figure 4.1. This similarity confirms that both estimators operated on the same underlying recapture data, with values starting at zero (event 1), peaking at 11 (event 2), declining to a minimum of 6 (event 6), and recovering modestly to 8 (event 7). The consistency in recapture patterns across methods allows for direct comparison of their statistical performance.

Figure 3: Shows the Recapture value by Sampling Event of the Horowitz-Thompson Estimator

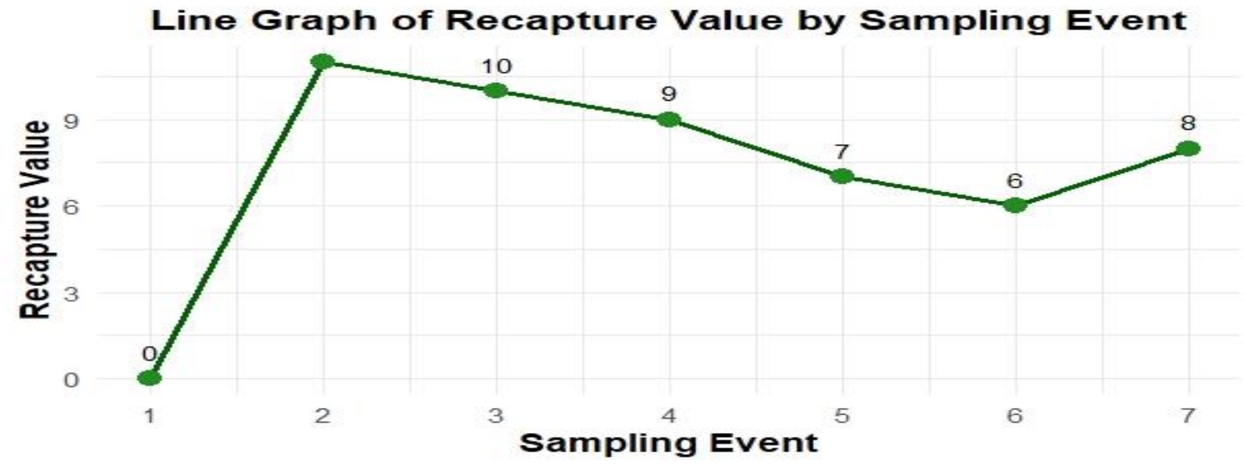

Figure 4 tracks the variance of the Horvitz-Thompson Estimator throughout the sampling events. Unlike the Peterson method's erratic variance (Figure 2), this estimator exhibits a smoother trajectory with lower overall variability, particularly during event 6, where variance peaks at 15.0, significantly less than Peterson’s 35.0. The comparatively subdued fluctuations demonstrate the method’s enhanced stability in handling sparse recapture data.

Figure 4: Shows the Variance of the Estimator by Sampling Event of the Horowitz-Thompson Estimator

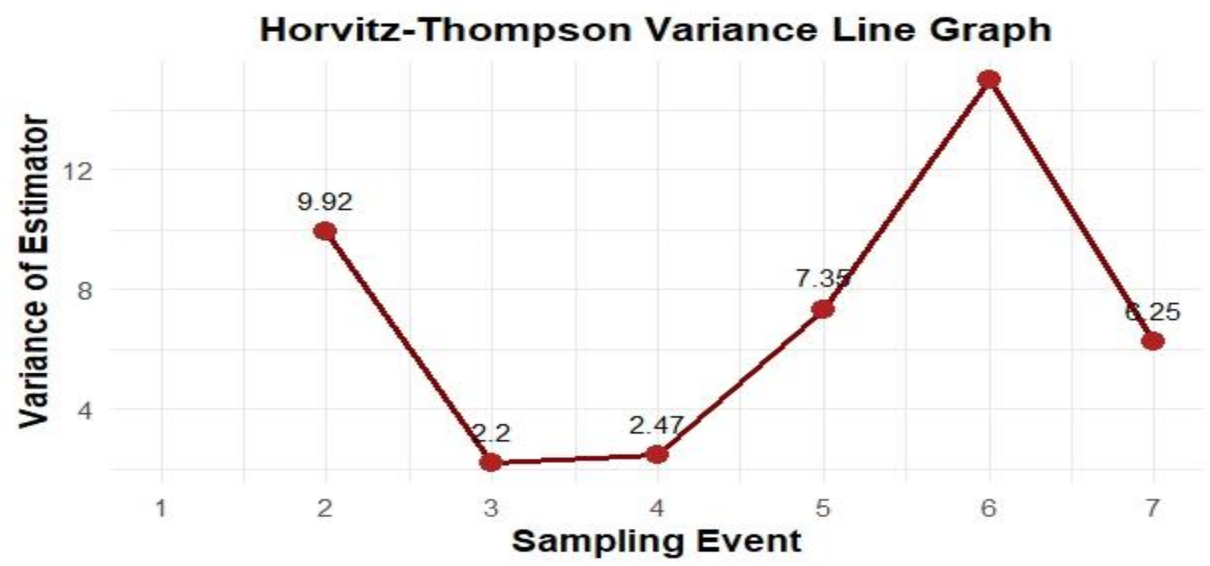

Table 3: Show Comparison of Horvitz-Thompson Estimator And Paterson Estimators

Sampling Event	Recapture	Peterson Estimate	Horvitz Estimate	Peterson Variance	Horvitz Variance
1	0	0.000	—	0.000	—
2	11	27.272	27.27	8.114	9.92
3	10	22.000	22.00	2.200	2.20
4	9	22.220	22.22	3.017	2.47
5	7	25.714	25.71	13.644	7.35
6	6	30.000	30.00	35.000	15.00
7	8	25.000	25.00	9.375	6.25

Figure 7 compares the population estimates of female drivers generated by the Peterson (dashed line) and Horvitz-Thompson (solid line) estimators across seven sampling events. Both methods produce nearly identical numerical results for each event, as evidenced by the overlapping trajectories that fluctuate between 22 and 30 after the initial zero estimate in event 1. This alignment demonstrates strong agreement in point estimates despite differing methodological approaches, where Peterson relies on simple recapture ratios and Horvitz-Thompson incorporates individual inclusion probabilities. The convergence highlights that both estimators yield consistent population size predictions under these sampling conditions, though their variance profiles (shown in Figure 7) reveal differences in precision.

Figure 5: Population size predictions under these sampling conditions through variance profiles

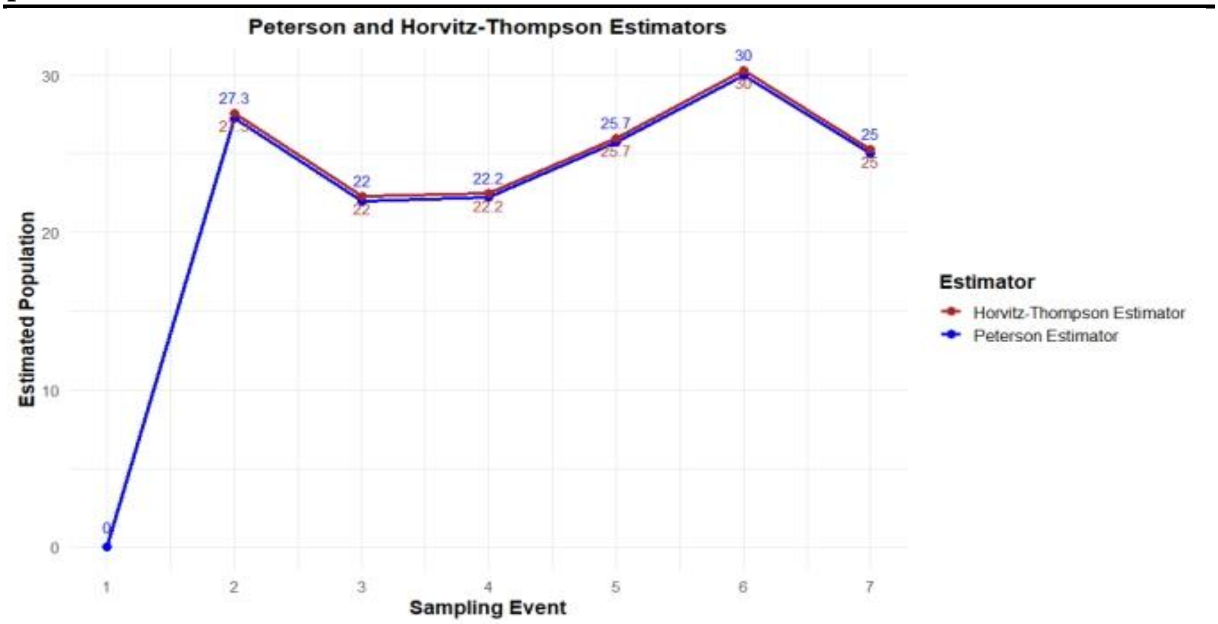

Table 3 presents a comparative evaluation of the Peterson and Horvitz-Thompson estimators across seven sampling events, with emphasis on the estimated population values and their corresponding variances. As shown in Table 3, both estimators produced nearly identical point estimates for events 2 through 7, indicating strong agreement in the computed population sizes of female drivers at BZU. The estimation values ranged from 22.00 to 30.00 in both methods, suggesting that under the given sampling conditions, where the number of recaptured individuals was moderate, the estimators are largely consistent in measuring central tendency. However, a closer inspection of the associated variances reveals meaningful differences between the two approaches, particularly in their sensitivity to sample variability and recapture frequency. In general, the Horvitz-Thompson estimator demonstrated lower variance compared to the Peterson estimator in most sampling events, indicating greater precision. For example, in event 6, both estimators returned a population size of 30.00. However, the variance of the Peterson estimator was 35.000, whereas the Horvitz-Thompson estimator yielded a substantially smaller variance of 15.000. The same situation was true in event 5 when the Peterson method registered a variance of 13.644 as opposed to 7.35 registered in Horvitz-Thompson. The differences matter since the smaller variance means having greater confidence in the estimate and a greater resistance to random sampling error. This strength is particularly evident in field research, where capture rates are not equal, and the samples might be limited.

This increase is explained by the fact that the methodological power of the Horvitz-Thompson estimator is that it directly uses the inclusion probabilities in the estimation itself. This enables it to respond better to unequal probability of sampling, and so is less exposed to biases or overestimation of variance where recaptures are rare. Conversely, the Peterson estimator presupposes a homogeneous probability of capturing both individuals and events, which could not be applicable in real life. The same assumption would have the ability to have an overestimate of variance, especially where there is low recapture, in events 5 and 6, though it did not happen. Despite the variability between the two methods, the actual estimate of both methods was nearly the same, estimating the events that occurred. This uniformity implies that the two estimators can provide accurate central estimates in the case of a rather balanced condition of the samples. Nevertheless, the Horvitz-Thompson estimator has a definite advantage that lies in the aspect of statistical efficiency, where it is measured in terms of variance and standard error. It has more reliable outcomes, narrower confidence intervals, and is more resistant to unreliable changes in recapture counts.

In conclusion, while both Peterson and Horvitz-Thompson estimators are effective tools for population estimation in capture-recapture studies, the Horvitz-Thompson method demonstrates superior performance in variance reduction and estimation precision. This makes it particularly suitable for practical applications where sampling irregularities and non-uniform capture probabilities are likely. The comparative results from Table 4.3 reinforce the preference for Horvitz-Thompson in settings where robustness and reliability are critical, even though both estimators produce comparable point estimates under similar data conditions.

Figure 6: Shows Comparison of Horvitz-Thompson Estimator And Paterson Estimators

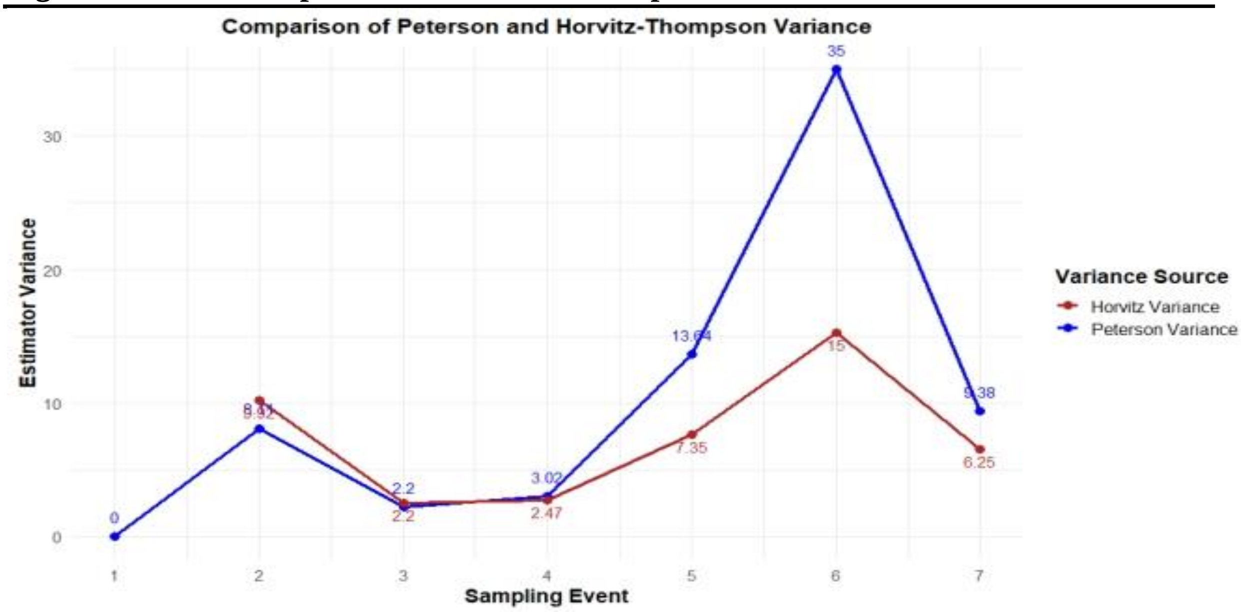

Figure 9 compares the variance (measurement uncertainty) of the Peterson and Horvitz-Thompson population estimators across seven sampling events. The Peterson estimator (dashed line) exhibits significantly higher and more volatile variance, peaking sharply at 35.0 in event 6. In contrast, the Horvitz-Thompson estimator (solid line) maintains consistently lower variance, with its highest value only reaching 15.0 in the same event. This divergence highlights the Horvitz-Thompson method's superior robustness, particularly when recapture rates are low or uneven, as it accounts for individual inclusion probabilities (πi), thereby reducing estimation error and providing a more reliable confidence interval.

Conclusion

This comparative study highlights the respective strengths and limitations of the Peterson and Horvitz-Thompson estimators in sparse recapture settings. While both methods produced consistent population estimates, the Horvitz-Thompson estimator proved statistically superior due to its lower variance and better precision. These results theoretically justify the preference for inclusion-probability-based estimators under non-uniform capture scenarios. Practitioners should therefore select estimators based on data conditions favoring Horvitz-Thompson when facing sampling heterogeneity.

Recommendations

1. Field researchers should consider estimator sensitivity to recapture variability when designing sampling strategies.
2. Future studies may incorporate Bayesian or hierarchical models to further improve precision.
3. Integration of modern data collection tools (e.g., automated recognition, mobile tracking) could enhance sampling accuracy.

These findings affirm that although the Peterson estimator remains useful for quick and approximate estimation under ideal conditions, the Horvitz-Thompson method offers a more robust and statistically sound alternative, particularly in complex, real-world sampling scenarios where capture probabilities may be uneven. The study highlights the importance of selecting estimation techniques that align with the nature of the data and field constraints.
Future research can expand this study by applying the estimators to larger and more diverse populations, incorporating modern data collection technologies such as video recognition or mobile tracking to improve accuracy. Additionally, exploring advanced statistical models such as Bayesian or hierarchical capture-recapture methods could further enhance estimation under variable field conditions. Simulation-based studies and behavioral modeling are also recommended to assess estimator performance across a broader range of scenarios with controlled parameters.

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