Robust Estimation of Variance Using Supplementary Information

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Abstract

This study introduces robust exponential ratio-type estimators for finite population variance, incorporating the minimum covariance determinant (M.C.D.) and minimum volume ellipsoid (M.V.E.) robust covariance matrices in the context of simple random sampling. We utilize a first-order approximation to derive expressions for the bias and mean square error (M.S.E.) of these proposed estimators. Through a comprehensive review of existing literature, it is evident that these robust estimators outperform other alternatives in terms of efficiency. Both the M.C.D. and M.V.E. methods are known for their resilience to outliers, making them particularly effective when such anomalies are present in the data. Simulation studies and empirical results consistently demonstrate that the proposed robust exponential ratio-type estimators yield lower mean square errors compared to traditional estimators under simple random sampling. Additionally, the efficiency of these estimators is further validated through simulated studies and real-world data sets. Theoretical analysis and numerical evidence collectively affirm that the proposed class of estimators consistently outperforms competing methods across various scenarios.

Keywords: Ratio-type Estimators; M.C.D; M.V.E; Population Variance; M.S.E; SRS.

Introduction

In numerous fields like business, agriculture, medicine, and biological sciences, where skewed populations are common, estimating the finite population variance is crucial. We encounter variation in every aspect of our daily lives. The natural law states that no two objects or people are exactly alike. In this regard, in order to prescribe medication appropriately, a doctor must have a thorough awareness of the variations in human blood pressure, body temperature, and pulse rate. A manufacturer cannot decide whether to lower or raise his prices or to enhance the quality of his product unless he is always aware of the degree of variety in people's reactions to it. Should schedule the planting of his crop in terms of time, location, and method, an agriculturist needs a sufficient understanding of differences in climate conditions, particularly from place to location (or from time to time). Researchers are occasionally more interested in the variance of their goods or outputs in manufacturing companies and pharmaceutical laboratories. In practical applications,

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there are numerous scenarios when estimating the population variance of the studied variable becomes crucial. For these reasons, a general populace variation estimate has drawn the attention of several authors. Supplemental data on the limited population being studied is frequently obtainable for sampling surveys through administrative databases, census data, or prior experience. There are many different ways to use auxiliary information to enhance the sampling design and provide more effective estimators of finite population variance, as described in the sampling literature. When the entire populace that selection comes to drawn is symmetrical, simple at-random selection will be used. Several writers have studied ratio-kind limited variation in population estimations utilizing information from additional factors. Das and Tripathi (1978) created a few estimation techniques of the ratio kind by calculating a finite variation in the populace while accounting for supplemental data. Isaki (1983) presented estimations of variance in different, for instance, configurations. Singh, Upadhyaya, and Namjoshi (1988) produced a pair of demographic variation estimate categories utilizing additional details. Agrawal and Sthapit (1995) presented a few fresh kinds of ratio methods that are based on additional data to calculate the population variance. Cebrián and García(1997) suggested estimating variation with additional details: The extremely impartial multivariable estimation of ratios. Arcos et al. (2005) developed a group that contains populace variation estimation techniques. Kadilar and Cingi (2006) recommended several populace-average calculated using ratios regarding basic as well as multilayered randomization. Shabbir and Gupta (2007) offered a unique ratio-based exponential estimate of the populace variation within the basic-random selection. Khan and Shabbir (2013) gave instructions for a population variance estimation on the ratio type that uses intervals regarding a supporting parameter. Subramani and Kumarapandiyan (2013) presented population variance estimations of the ratio type based on the median value and known coefficient of variation of an auxiliary variable. Sharma and Singh (2013) suggested a generalized class of estimators that can be used with respect to limited variation in populations when measuring inaccuracies are present. Singh, Pal, and Solanki (2014) introduced a novel family for population variance estimation. Solanki, Singh, and Pal (2015) offered a few estimation methods of limited variation in populations in the ratio-type based on quartiles or other auxiliary variables with specified values for the parameters. Yadav et al. (2015) developed a family of population variance estimators in simple random sampling that is better. Singh and Pal (2016) introduced a group for finite population variance estimators. Yaqub and Shabbir (2016) proposed a new class of estimators regarding variance in a finite population. Sanaullah, Asghar, and Hanif (2017) offered a group for estimations using populace variation estimation together with a generalized exponential estimator. Shahzad et al. (2017) offered estimations for a limiting populace using ratios form Singh and Pal (2017) suggested using the known coefficient of variation for an auxiliary variable within sample surveys to estimate the population variance. Sanaullah et al. (2017) developed a broad class of exponential estimators that can be used to calculate limited populace variance. Shahzad et al. (2018) introduced a novel family among estimations that can be used to calculate the limited variability of individuals regardless of an additional characteristic within a failure-to-react issue. Sharma et al. (2018) proposed estimators for population variance using auxiliary information on quartiles in simple random sampling. Hanif and Shahzad (2019) introduced ratio estimators for simple random sampling, utilizing the trace of the kernel matrix to estimate population variance in scenarios without non-response. Abid et al. (2019) proposed a general class of estimators that leverage information from the midrange and inter-decile range of an auxiliary variable to estimate population variance. Singh and Khalid (2019) tackled the challenge of estimating variation in current samples under conditions of randomized non-response in two-occasion sequential selection with a dual-phase approach. Zaman and Bulut (2019) developed estimators based on minimum covariance determinant (M.C.D.) estimates. Kumari and Thakur (2020) introduced an advanced class of log-type estimators for population variance that incorporate both attribute and variable data. In a subsequent study, Zaman and Bulut (2020) proposed ratio estimators that consider a strong correlation matrix in the presence of outliers. Daraz and Khan (2021) presented a class of estimators aimed at estimating the population variance of the variable of interest. Shahzad et al. (2021) utilized L-moments to develop a novel class of estimators for finite population variance, particularly effective in the presence of extreme values. Their work also included estimators designed to mitigate the adverse effects of anomalies. Ahmad et al. (2022) introduced an enhanced population variance estimator using dual auxiliary data under simple random sampling. Bhushan et al. (2022) proposed efficient classes of estimators for population variance, incorporating attributes within a simple random sampling framework. Zaman and Bulut (2022) advanced an improved estimator for finite population variance estimation using dual auxiliary variables under stratified random sampling. Ahmad et al. (2023) extended this work by further enhancing the estimator for finite population variance using dual auxiliary variables, again under stratified random sampling. Finally, Kumar and Choudhary (2023) introduced a ratio-cum-exponential estimator for finite population variance, employing two auxiliary variables in simple random sampling. Mittal and Kumar (2023) introduced two new estimators of finite population variance under random non-response in simple random sampling. Zaman and Bulut(2023) suggested a resilient estimate of constrained variability in populations using a ratio-type approach while taking the robust correlation matrices in simple as well stratified random sampling. Shahzad et al. (2023) introduced using well-known descriptions of a supplementary parameter, an entirely novel expanded group of robustness sort of variation estimations is suggested. Alomair and Gardazi (2024) introduced, by using well-known descriptions of a supplementary parameter, an innovative extended category of resilient sort of variation estimations. Qureshi et al. (2024) suggested memory-type ratios and goods estimations for variation in populations in time-scaled surveying with infinitely biased average movements.

The rest of the article is organized as follows: In Section 2, notation is shown, and also, in sub-section 2.1, existing estimators are shown. In Section 3, the robust covariance matrix is shown and also in sub-sections 3.1.1 and 3.1.2, an existing estimator of population variance under simple random sampling is shown. In Section 4, the proposed variance estimators using the MCD and the M.V.E. estimates are developed. In Section 5, a numerical study is conducted. In Section 6, a simulation study is conducted. The article concludes in section 7.

Notations

Consider a finite population $u=(u_1,u_2,....,u_N)$ having N units. Let y and x are study, and auxiliary variable. We will use the following notations in this article explained aforementioned:

$\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$, $\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$:Sample means of auxiliary variable x and study variable y respectively,

$\bar{X}=\frac{1}{N}\sum_{i=1}^{N}X_i$, $\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}Y_i$: Population means of auxiliary variable x and study variable y respectively,

$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$, $s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$: Sample variance of auxiliary variable x and study variable y respectively,

$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2$, $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)^2$: Population mean square/variance of the auxiliary variable x and study variable y,

$\lambda_{04} = \frac{\mu_{04}}{\mu_{02}^2}$, $\lambda_{40} = \frac{\mu_{40}}{\mu_{20}^2}$: Coefficient of Kurtosis of auxiliary variable x and study variable y,

Where $\lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}}$ and $\mu_{rs} = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^r\left(x_i - \bar{X}\right)^s$, r and s are non-negative integers.

Existing estimators

We go over each of the upcoming estimation and demonstrate which ones perform better in certain scenarios. Assumedly, the population variation S_x^2 provided in this investigation for the supplementary variable x

The variance of the unbiased estimator $\left(s_0 = s_y^2\right)$ as

$$V\left(s_0\right) = \frac{S_y^4}{n}\left(\lambda_{40} - 1\right) \tag{2.1}$$

If the population variance S_y^2 of the auxiliary variable x is known, Isaki (1983) introduced the ratio estimator for s_y^2 as

$$s_r = s_y^2 \frac{S_x^2}{s_x^2} \tag{2.2}$$

The estimator's mean square error s_r, it is provided by

$$MSE\left(s_r\right) = \frac{S_y^4}{n}\left[\left(\lambda_{40} - 1\right) + \left(\lambda_{04} - 1\right)\left(1 - 2C\right)\right] \tag{2.3}$$

Examining (2.1) and (2.3), Isaki's (1983) the estimation method yields a smaller mean square error (M.S.E) compared to the independent estimation when $C > 0.5$

In the occurrence that the population variation S_x^2 of the supplementary variable x is understood, as well as the s_y^2 in (2.2) is replaced with s_r then Singh et al. (2018) offered rise to the chain ratios estimation as

$$s_{cr} = s_r \frac{S_x^2}{s_x^2} \tag{2.4}$$

We can rewrite (2.4) using (2.2) as

$$s_{cr} = s_y^2 \frac{S_x^4}{s_x^4} \tag{2.5}$$

The estimator's mean square error s_r, provided by

$$MSE\left(s_{cr}\right)=\frac{S_y^4}{n}\left[\left(\lambda_{40}-1\right)+\left(\lambda_{04}-1\right)\left(1-C\right)\right] \quad (2.6)$$

Where, $C=\frac{\left(\lambda_{22}-1\right)}{\left(\lambda_{04}-1\right)}$

Examining (2.1) and (2.6), Singh's et al. (2018) the estimation method yields a smaller mean square error (M.S.E) compared to the independent estimation when $C>1$. From (2.3) and (2.6), Singh's et al. (2018) estimator offers an M.S.E that is less than the Isaki's (1983) estimate in the given scenario $C>1.5$.

In case of the populace variation S_x^2 of the additional x variable is understood, Isaki (1983) defined the following regression estimator for S_y^2, given by

$$s_{reg}=s_y^2+b\left(S_x^2-s_x^2\right) \quad (2.7)$$

When sample regression coefficient (b) is given.

The estimator's mean square error s_r, provided by

$$MSE\left(s_{reg}\right)=\frac{S_y^4}{n}\left(\lambda_{40}-1\right)\left(1-\rho\right) \quad (2.8)$$

Where $\rho=\frac{\left(\lambda_{22}-1\right)}{\sqrt{\left(\lambda_{40}-1\right)\left(\lambda_{04}-1\right)}}$

Examining (2.1) and (2.8), Isaki's (1983) when the following conditions are met, the regression-based ratio-type estimate yields less M.S.E compared to the unassisted estimate $\rho^2>0$, due to the fact that the requirement is constantly met

In cases where the populace variation S_x^2 of the auxiliary variable x is known, Upadhyaya and Singh (1999) introduced the ratio estimator for S_y^2 as

$$s_{us}=s_y^2\left(\frac{S_x^2+\lambda_{04}}{s_x^2+\lambda_{04}}\right) \quad (2.9)$$

The estimator's mean square error s_{us}, provided by

$$MSE\left(s_{us}\right)\cong\left[\left(\lambda_{40}-1\right)+g_0^2\left(\lambda_{04}-1\right)-2g_0\left(\lambda_{22}-1\right)\right] \quad (2.10)$$

Where $g_0=\frac{S_x^2}{S_x^2+\lambda_{04}}$.

Examining (2.1) and (2.10), Upadhyaya and Singh's (1999) the estimation method yields a smaller mean square error (M.S.E) compared to the independent estimation when $g_0<\frac{2\left(\lambda_{22}-1\right)}{\left(\lambda_{04}-1\right)}$.

When the population variance S_x^2 of the auxiliary variable x is known, Kadilar and Cingi (2006) provided the ratio estimator for S_y^2 as

$$s_{kc1}=s_y^2\left(\frac{S_x^2+C_x}{s_x^2+C_x}\right) \quad (2.11)$$

$$s_{kc2} = s_y^2 \left(\frac{\lambda_{04} S_x^2 + C_x}{\lambda_{04} s_x^2 + C_x} \right) \tag{2.12}$$

$$s_{kc3} = s_y^2 \left(\frac{C_x S_x^2 + \lambda_{04}}{C_x s_x^2 + \lambda_{04}} \right) \tag{2.13}$$

where $C_x = \frac{S_x}{\bar{X}}$ is the population coefficient of variation.

The estimator's mean square error $s_{kci}\left(i = 1,2,3\right)$ are provided below

$$MSE\left(s_{kci}\right) \cong \frac{S_y^4}{n}\left[\left(\lambda_{40} - 1\right) + g_i^2\left(\lambda_{04} - 1\right) - 2g_i\left(\lambda_{22} - 1\right)\right] \tag{2.14}$$

Where $g_1 = \frac{S_x^2}{S_x^2 + C_x}$, $g_2 = \frac{\lambda_{04} S_x^2}{\lambda_{04} S_x^2 + C_x}$, $g_3 = \frac{C_x S_x^2}{C_x S_x^2 + \lambda_{04}}$.

Examining (2.1) and (2,14), Kadilar and Cingi's (2006) the estimation method yields a smaller mean square error (M.S.E) compared to the independent estimation when $g_i < \frac{2\left(\lambda_{22} - 1\right)}{\left(\lambda_{04} - 1\right)}, i = 1,2,3.$

From (2.10) and (2.14), Kadilar and Cingi's (2006) estimations offer a mean square error that is less than the Upadhyaya and Singh's (1999) estimate in this particular scenario $g_i < \frac{2\left(\lambda_{22} - 1\right)}{\left(\lambda_{04} - 1\right)} - g_0, i = 1,2,3.$

Robust covariance matrix (Minimum covariance determinant (M.C.D) and Minimum volume ellipsoid (M.V.E) estimator)

The M.C.D estimator for location parameter of multivariate data was defined as by Rousseeuw (1985);

$T(X)$=mean of the h point of X for which the determinant of covariance matrix is minimal (3.1)

Whereas α is trimming ratio and $h = (1-\alpha)*n$. The covariance matrix of this subset is the M.C.D estimator of the scatter parameter. Furthermore, the trimming ratio is equivalent to the breakdown point of M.C.D estimators. If $h = 0.5*n$, the maximum breakdown point of the estimator is 50%. Nevertheless, the balance between robustness and efficiency, generally, h is defined as $h = 0.75*n$ and breakdown point is 25% (Rousseeuw & van Driessen 1999; Bulut & Oner 2017; Bulut, 2014).

The M.V.E estimator for location parameter of multivariate data was defined

$T(X)$=center of an ellipsoid having minimum volume spanned by the h points in X data. (3.2)

For details, see (Rousseeuw 1985; Bulut 2014).

The Existing Estimator

In this Section, Zaman and Bulut (2022) presented population variance estimators equivalent to regression for basic random sampling. Yet whenever there is an anomaly in the data set, these traditional estimators become less effective. Thus, robust ratio-type estimates have been added to the classical ratio estimators that were first provided to remove the outlier problem's detrimental effects in subsection 3.1.2.

The Existing Regression-Type Estimators

For the population variance, Zaman and Bulut (2022) provided the following regression-ratio-type estimator s_y^2 as

$$s_{rZB} = \frac{s_y^2 + b\left(S_x^2 - s_x^2\right)}{\left(\zeta_1 s_x^2 + \zeta_2\right)}\left(\zeta_1 S_x^2 + \zeta_2\right) \tag{3.3}$$

Where ζ_1 and ζ_2 are either constants or variables of the auxiliary variable's parameters such as C_x, $\beta_{2(x)}$ as well ρ. To determine the M.S.E of the estimator in (3.1), the terms with e's are defined as follows:

Let $e_0 = \left(s_y^2 - S_y^2\right)/S_y^2$, and $\left(s_x^2 - S_x^2\right)/S_x^2$, such that $E\left(e_I\right) = 0$, $I = 0,1$. $E\left(e_0^2\right) = \frac{\left(\lambda_{40} - 1\right)}{n}$, $E\left(e_1^2\right) = \frac{\left(\lambda_{04} - 1\right)}{n}$, and $E\left(e_0 e_1\right) = \frac{\left(\lambda_{22} - 1\right)}{n}$.

Following Singh and Malik (2014), the S_{zbi}^2 in term of e's, we have

$$s_{rZB} = \left[S_y^2\left(1 + e_0\right) - bS_x^2 e_1\right]\left[1 + a_i e_1^2\right]^{-1}$$

The formulations of M.S.E are accurate to the first order of approximation for s_{rZB} is provided by

$$\left(S_{zbi}^2 - S_y^2\right) = \left(S_y^2 e_o - bS_x^2 e_1 - a_i S_y^2 e_1\right)^2$$

$$\left(s_{rZB} - S_y^2\right)^2 \cong \left[S_y^4 e_0^2 + \left(b^2 S_x^4 + a_i^2 S_y^4 + 2a_i b S_y^2 S_x^2\right)e_1^2 - 2S_y^2 e_0 e_1\left(bS_x^2 + a_i S_y^2\right)\right]$$

$$MSE\left(s_{rZB}\right) \cong \frac{1}{n}\left\{S_y^4\left(\lambda_{40} - 1\right) + \left(\lambda_{04} - 1\right)\left(B^2 S_x^4 + a_1^2 S_y^4 + 2a_i B S_y^2 S_x^2\right) - 2S_y^2\left(\lambda_{22} - 1\right)\left[BS_x^2 + a_1 S_y^2\right]\right\} \tag{3.4}$$

Where $a_1 = \frac{\zeta_1 S_x^2}{\zeta_1 S_x^2 + \zeta_2}$.

Table 1: Existing estimators (Equation (3.3))

Estimators	Values of	
	ζ_1	ζ_2
$s_{rZB1} = \frac{s_y^2 + b\left(S_x^2 - s_x^2\right)}{s_x^2} S_x^2$	1	0
$s_{rZB2} = \frac{s_y^2 + b\left(S_x^2 - s_x^2\right)}{\left(s_x^2 + \beta_2(x)\right)}\left(S_x^2 + \beta_2(x)\right)$	1	$\beta_2(x)$
$s_{rZB3} = \frac{s_y^2 + b\left(S_x^2 - s_x^2\right)}{\left(s_x^2 + C_x\right)}\left(S_x^2 + C_x\right)$	1	C_x

$s_{rZB4}=\dfrac{s_y^2+b\left(S_x^2-s_x^2\right)}{\left(s_x^2+\rho\right)}\left(S_x^2+\rho\right)$	1	ρ
$s_{rZB5}=\dfrac{s_y^2+b\left(S_x^2-s_x^2\right)}{\left(s_x^2\beta_2(x)+C_x\right)}\left(S_x^2\beta_2(x)+C_x\right)$	$\beta_2(x)$	C_x
$s_{rZB6}=\dfrac{s_y^2+b\left(S_x^2-s_x^2\right)}{\left(s_x^2C_x+\beta_2(x)\right)}\left(S_x^2C_x+\beta_2(x)\right)$	C_x	$\beta_2(x)$
$s_{rZB7}=\dfrac{s_y^2+b\left(S_x^2-s_x^2\right)}{\left(s_x^2C_x+\rho\right)}\left(S_x^2C_x+\rho\right)$	C_x	ρ
$s_{rZB8}=\dfrac{s_y^2+b\left(S_x^2-s_x^2\right)}{\left(s_x^2\rho+C_x\right)}\left(S_x^2\rho+C_x\right)$	ρ	C_x
$s_{rZB9}=\dfrac{s_y^2+b\left(S_x^2-s_x^2\right)}{\left(s_x^2\beta_2(x)+\rho\right)}\left(S_x^2\beta_2(x)+\rho\right)$	$\beta_2(x)$	ρ
$s_{rZB10}=\dfrac{s_y^2+b\left(S_x^2-s_x^2\right)}{\left(s_x^2\rho+\beta_2(x)\right)}\left(S_x^2\rho+\beta_2(x)\right)$	ρ	$\beta_2(x)$

The Existing Class of Robust Estimators

Zaman and Bulut (2022) define to apply the following ratio estimators for the population variance s_y^2 using robust covariance estimates to data which have outliers.

$$s_{rZB(j)}=\frac{s_{y(j)}^2+b_{(j)}\left(S_{x(j)}^2-s_{x(j)}^2\right)}{\left(\zeta_{1(j)}s_{x(j)}^2+\zeta_{2(j)}\right)}\left(\zeta_{1(j)}S_{x(j)}^2+\zeta_{2(j)}\right) \tag{3.5}$$

where $s_{y(j)}^2$, $b_{(j)}$, $S_{x(j)}^2$, $s_{x(j)}^2$, $\zeta_{1(j)}$ and $\zeta_{2(j)}$ are acquired by taking into account M.V.E and M.C.D covariance estimates, respectively.

Using (3.3), The following M.S.E is derived for each of the existing estimators that correspond to the relevant robust covariance estimates:

$$MSE\left(s_{rZB(j)}\right)\cong\frac{1}{n}\left\{S_{y(j)}^4\left(\lambda_{40(j)}-1\right)+\left(\lambda_{04(j)}-1\right)\left[B_{(j)}^2S_{x(j)}^4+a_{1(j)}^2S_{y(j)}^4+2a_{1(j)}B_{(j)}S_{y(j)}^2S_{x(j)}^2\right]-2S_{y(j)}^2\left(\lambda_{22(j)}-1\right)\left[B_{(j)}S_{x(j)}^2+a_{1(j)}S_{y(j)}^2\right]\right\}; \tag{3.6}$$

$j=MCD$ and MVE

It is important to note that the M.S.E equation for the existing group of resilient estimations is identical to the M.S.E formula given in (3.2), however, it is evident that S_y^4, λ_{40}, B, S_x^4, λ_{22}, and a_1 in (3.1) must be switched out for $S_{y(j)}^4$, $\lambda_{40(j)}$, $B_{(j)}$, $S_{x(j)}^4$, $\lambda_{22(j)}$, and $a_{1(j)}$, whose figures were determined using strong covariance estimations $\left(j=MCD \text{ and } MVE\right)$.

Table 2: Existing estimators (Equation (3.5))

Estimators	Values of	
	$\zeta_{1(j)}$	$\zeta_{2(j)}$
$s_{rZB1(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{s^2_{x(j)}} S^2_{x(j)}$	1	0
$s_{rZB2(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(s^2_{x(j)} + \beta_{2(j)}(x)\right)}\left(S^2_{x(j)} + \beta_{2(j)}(x)\right)$	1	$\beta_{2(j)}(x)$
$s_{rZB3(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(s^2_{x(j)} + C_{x(j)}\right)}\left(S^2_{x(j)} + C_{x(j)}\right)$	1	$C_{x(j)}$
$s_{rZB4(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(s^2_{x(j)} + \rho_{(j)}\right)}\left(S^2_{x(j)} + \rho_{(j)}\right)$	1	$\rho_{(j)}$
$s_{rZB5(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(s^2_{x(j)}\beta_{2(j)}(x) + C_{x(j)}\right)}\left(S^2_{x(j)}\beta_{2(j)}(x) + C_{x(j)}\right)$	$\beta_{2(j)}(x)$	$C_{x(j)}$
$s_{rZB6(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(s^2_{x(j)}C_{x(j)} + \beta_{2(j)}(x)\right)}\left(s^2_{x(j)}C_{x(j)} + \beta_{2(j)}(x)\right)$	$C_{x(j)}$	$\beta_{2(j)}(x)$
$s_{rZB7(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(S^2_{x(j)}C_{x(j)} + \rho_{(j)}\right)}\left(S^2_{x(j)}C_{x(j)} + \rho_{(j)}\right)$	$C_{x(j)}$	$\rho_{(j)}$
$s_{rZB8(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(S^2_{x(j)}\rho_{(j)} + C_{x(j)}\right)}\left(S^2_{x(j)}\rho_{(j)} + C_{x(j)}\right)$	$\rho_{(j)}$	$C_{x(j)}$
$s_{rZB9(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(S^2_{x(j)}\beta_{2(j)}(x) + \rho_{(j)}\right)}\left(S^2_{x(j)}\beta_{2(j)}(x) + \rho_{(j)}\right)$	$\beta_{2(j)}(x)$	$\rho_{(j)}$
$s_{rZB10(j)} = \frac{s^2_{y(j)} + b_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{\left(S^2_{x(j)}\rho_{(j)} + \beta_{2(j)}(x)\right)}\left(S^2_{x(j)}\rho_{(j)} + \beta_{2(j)}(x)\right)$	$\rho_{(j)}$	$\beta_{2(j)}(x)$

Proposed robust exponential ratio-type estimators

Taking motivation from Zaman and Bulut (2022), we first derive robust exponential ratio-type estimators for simple random sampling, After that, we examine how efficient these estimators are in comparison to current ones.

Robust exponential ratio-type estimators of finite population variance in simple random sampling

As previously noted, estimators that are founded on the traditionalclassical covariance matrix are employed in literary worksto estimate the population variance. In this work, we suggested eight exponential ratio-type estimators with the use of M.C.D and eight exponential ratio-type estimators utilizing M.V.E covariance estimators to datacontaining anomalies. Below is how these estimators have been altered:

$$S^2_{PR1(j)} = \left[s^2_{y(j)} + \hat{\beta}_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)\right]\exp\left[\frac{p_{0.15}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{p_{(0.15)}\left(S^2_{x(j)} + s^2_{x(j)} + 2D_1\right)}\right] \tag{4.1}$$

$$S^2_{PR2(j)} = \left[s^2_{y(j)} + \hat{\beta}_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)\right]\exp\left[\frac{p_{(0.25)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{p_{(0.25)}\left(S^2_{x(j)} + s^2_{x(j)} + 2D_2\right)}\right] \tag{4.2}$$

$$S^2_{PR3(j)} = \left[s^2_{y(j)} + \hat{\beta}_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)\right]\exp\left[\frac{p_{(0.35)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{p_{(0.35)}\left(S^2_{x(j)} + s^2_{x(j)} + 2D_3\right)}\right] \tag{4.3}$$

$$S^2_{PR4(j)} = \left[s^2_{y(j)} + \hat{\beta}_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)\right]\exp\left[\frac{p_{(0.45)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{p_{(0.45)}\left(S^2_{x(j)} + s^2_{x(j)} + 2D_4\right)}\right] \tag{4.4}$$

$$S^2_{PR5(j)} = \left[s^2_{y(j)} + \hat{\beta}_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)\right]\exp\left[\frac{p_{(0.55)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{p_{(0.55)}\left(S^2_{x(j)} + s^2_{x(j)} + 2D_5\right)}\right] \tag{4.5}$$

$$S^2_{PR6(j)} = \left[s^2_{y(j)} + \hat{\beta}_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)\right]\exp\left[\frac{p_{(0.65)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{p_{(0.65)}\left(S^2_{x(j)} + s^2_{x(j)} + 2D_6\right)}\right] \tag{4.6}$$

$$S^2_{PR7(j)} = \left[s^2_{y(j)} + \hat{\beta}_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)\right]\exp\left[\frac{p_{(0.75)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{p_{(0.75)}\left(S^2_{x(j)} + s^2_{x(j)} + 2D_7\right)}\right] \tag{4.7}$$

$$S^2_{PR8(j)} = \left[s^2_{y(j)} + \hat{\beta}_{(j)}\left(S^2_{x(j)} - s^2_{x(j)}\right)\right]\exp\left[\frac{p_{(0.85)}\left(S^2_{x(j)} - s^2_{x(j)}\right)}{p_{(0.85)}\left(S^2_{x(j)} + s^2_{x(j)} + 2D_8\right)}\right] \tag{4.8}$$

It is important to note that percentiles $\left(p_i = 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85\right)$, and deciles $\left(D_i = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\right)$ will be only calculated for X variable not from M.C.D and M.V.E. $s^2_{y(j)}$, $\hat{\beta}_{(j)}$, $S^2_{x(j)}$, $s^2_{x(j)}$, are acquired by taking into account M.V.E and M.C.D covariance estimates, respectively. $\hat{\beta}$ is the sample regression coefficient.

To get the M.S.Es of the estimators $S^2_{PRi(j)}$ up-to first order of approximations are derived under the following transformations:

Let $e_0 = \frac{s_y^2 - S_y^2}{S_y^2}$ and $e_1 = \frac{s_x^2 - S_x^2}{S_x^2}$, Such that $E(e_i) = 0$, $i = 0,1$. $E(e_0^2) = \frac{(\lambda_{40} - 1)}{n}$, $E(e_1^2) = \frac{(\lambda_{04} - 1)}{n}$, and $E(e_0 e_1) = \frac{(\lambda_{22} - 1)}{n}$. Take the following forms:

$$S_{PRi(j)}^2 = \left[S_{y(j)}^2 (1 + e_0) + \hat{\beta}_{(j)} \left(S_{x(j)}^2 - S_{x(j)}^2 (1 + e_1) \right) \right] \left(1 - \frac{\theta_{(j)} e_1}{2} + \frac{3\theta_{(j)}^2 e_1^2}{8} \right) \tag{4.9}$$

$$S_{PRi(j)}^2 - S_{y(j)}^2 = S_{y(j)}^2 \left[e_0 - \frac{1}{2} \left(\theta_{(j)} e_1 \right) - \beta_{(j)} \frac{S_{x(j)}^2}{S_{y(j)}^2} e_1 \right] \tag{4.10}$$

Whereas $\theta_{(j)} = \frac{p_{(i)} S_{x(j)}^2}{p_{(i)} S_{x(j)}^2 + D_{(i)}}$ and by taking the square of the aforementioned formulas, taking expectation, and then simplifying them, we were able to find the following M.S.E equations. Therefore, the following are the M.S.Es of the suggested estimator:

$$MSE\left(S_{PRi(j)}^2\right) = \frac{1}{n} \Big\{ S_{y(j)}^4 \left(\lambda_{40(j)} - 1 \right) + S_{x(j)}^2 \left(\lambda_{04(j)} - 1 \right) \left[\frac{1}{4} A_{i(j)}^2 S_{y(j)}^2 + \beta_{(j)}^2 S_{x(j)}^2 + A_{i(j)} \beta_{(j)} S_{x(j)}^2 \right]$$
$$- S_{x(j)}^2 S_{y(j)}^2 \left(\lambda_{22(j)} - 1 \right) \left[A_{i(j)} + 2\beta_{(j)} \right] \tag{4.11}$$

$i = 1,2,3,4,5,6,7,8;$ $j = MCD$ and MVE

Where,

$A_{1(j)} = \frac{p_{(0.15)} S_{y(j)}^2}{p_{(0.15)} S_{x(j)}^2 + D_1}$, $A_{2(j)} = \frac{p_{(0.25)} S_{y(j)}^2}{p_{(0.25)} S_{x(j)}^2 + D_2}$, $A_{3(j)} = \frac{p_{(0.35)} S_{y(j)}^2}{p_{(0.35)} S_{x(j)}^2 + D_3}$, $A_{4(j)} = \frac{p_{(0.45)} S_{y(j)}^2}{p_{(0.45)} S_{x(j)}^2 + D_4}$,

$A_{5(j)} = \frac{p_{(0.55)} S_{y(j)}^2}{p_{(0.55)} S_{x(j)}^2 + D_5}$, $A_{6(j)} = \frac{p_{(0.65)} S_{y(j)}^2}{p_{(0.65)} S_{x(j)}^2 + D_6}$,

$A_{7(j)} = \frac{p_{(0.75)} S_{y(j)}^2}{p_{(0.75)} S_{x(j)}^2 + D_7}$, $A_{8(j)} = \frac{p_{(0.85)} S_{y(j)}^2}{p_{(0.85)} S_{x(j)}^2 + D_8}$, and $\beta_{(j)} = \frac{S_{y(j)}^2 \left(\lambda_{22(j)} - 1 \right)}{S_{x(j)}^2 \left(\lambda_{04(j)} - 1 \right)}$.

Numerical illustration

In order to have a route to the application of the suggested estimators, we frequently depend on empirical research. Because of this reason, We use two distinct population datasets. M.S.Es and percent relative efficiency will be used to draw a conclusion. We examine the population dataset for the purpose of assess the behavior among the suggested estimators, as it contains outliers.The population statistics from datasets A and B are shown in Tables 3 and 4. We calculated the mean square error (M.S.E) for both the existing and the suggested robust exponential ratio-type estimators. And determine the percent relative efficiency of the suggested robust exponential ratio-type estimators using these values, provided in Equations (4.1)-(4.8) regarding the existing estimators provided in Equation (3.3) and (3.5) by using the Equation (5.1) as below:

$$PRE\left(Existing\ \ Estimators, proposed\ \ Estimators\right) = \frac{MSE\left(Existing\ Estimator\right)}{MSE\left(Proposed\ Estimator\right)} \times 100 \quad (5.1)$$

Tables 5 and 6 from dataset A and tables 7 and 8 from dataset B contain these values.These relative efficiency values show that, in comparison to the existing estimators, all of the estimators based on M.C.D and M.V.E estimators have percentage relative efficiencies greater than 100. Furthermore, based on theoretical comparisons, we can see from Tables 5, 6, 7, 8 that, in data containing outliers, all of the suggested robust exponential ratio-type estimators based on M.C.D and M.V.E estimators have less M.S.E values than the existing estimators under simple random sampling.

Figure1:Scatterplot for Dataset-A

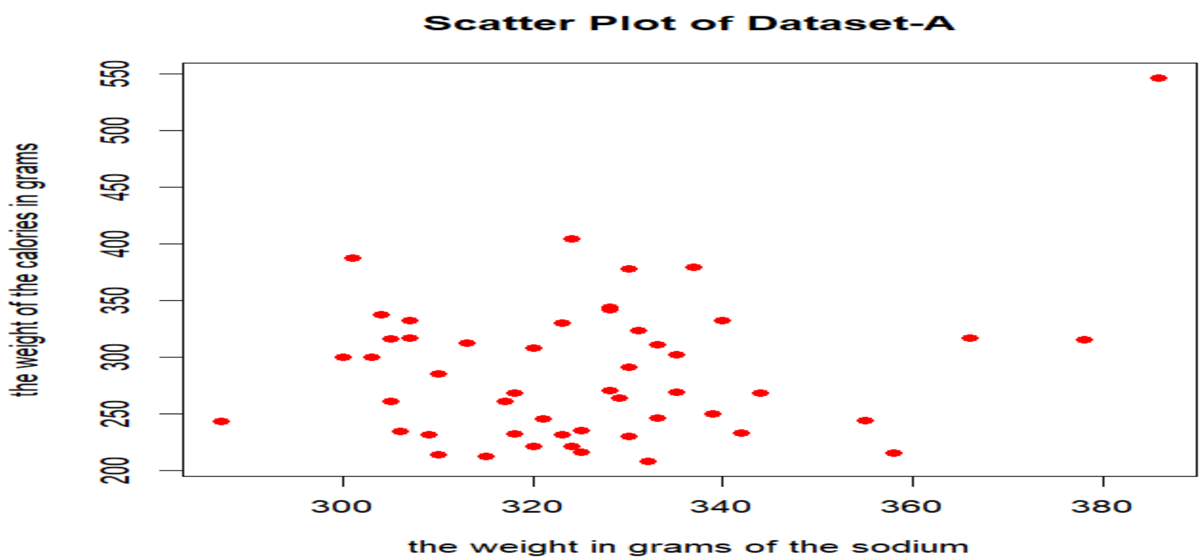
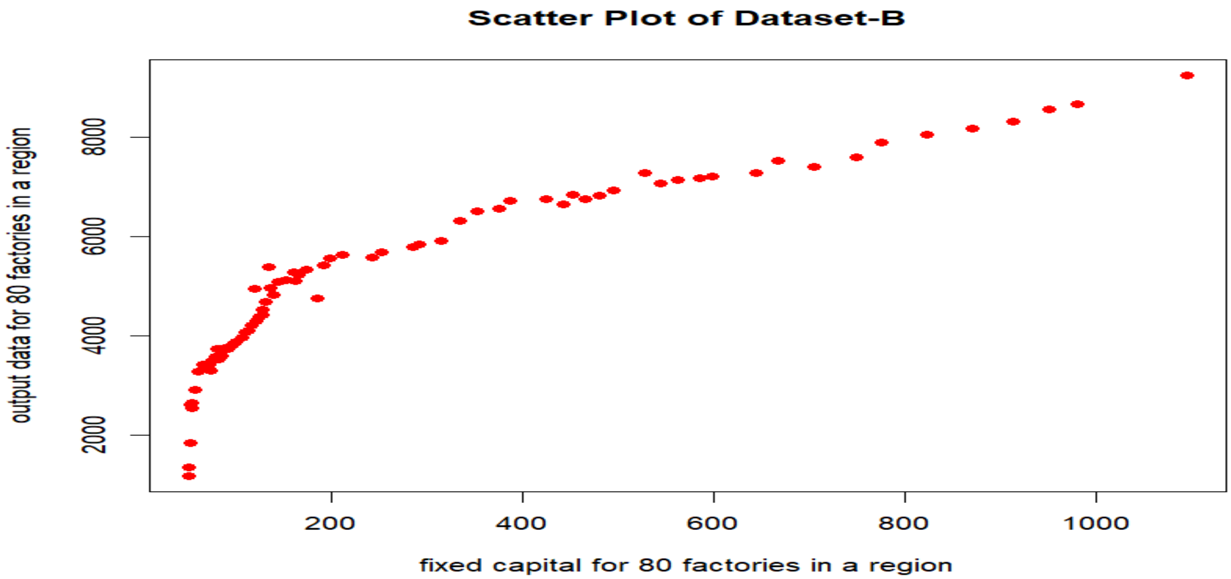


Figure 2: Scatterplot for Dataset-B



Uscereals Data (Dataset-A)

The Us-cereals (Ripley et al. 2013) dataset available in MASS package in R.The variables for dataset-A are:

X= the weight in grams of the sodium.

Y= the weight of the calories in grams.

Dataset-B

To demonstrate the performance of the proposed estimationapproach used in this work, we examine from the“Murthy” (page no: 288) (1976) where, we consider the following variables for dataset-B

X= Fixed capital for 80 factories in a region.

Y= Output data for 80 factories in a region.

Where, population size=80 and sample size=20.

Figures 1 and 2 clearly show outliers in the population, so it is reasonable to assume that the suggested estimator will outperform the one currently in use in the literature.

Table 3: Data statistics of dataset -A

	Classical	**M.C.D**	**M.V.E**	**Real values**
$\bar{Y}$	149.398	137.8431	130.9863	139.409
$\bar{X}$	273.8384	244.5698	244.5698	254.326
S_y^2	3895.306	4030.908	4239.595	4357.633
S_x^2	17064.1	17110.12	17110.12	17094.78
C_y	0.4177592	0.4605921	0.4970917	0.458481
C_x	0.549237	0.5348398	0.5348398	0.539638

$\lambda_{04}=\beta_{2(x)}$	8.065066	7.77692	7.77692	7.872968
$\lambda_{40}=\beta_{2(y)}$	10.09817	11.16163	11.1877	10.81583
λ_{22}	4.455478	4.545908	4.55644	4.519275
β	0.1116478	0.1232665	0.1300333	0.121649
A_1	1.00	0.2355767	0.2477728	0.494449
A_2	0.9995276	0.2355732	0.2477692	0.49429
A_3	0.9999678	0.2355733	0.2477694	0.494436
A_4	0.9999747	0.2355728	0.2477688	0.494438
A_5	0.999996	0.2355738	0.2477698	0.494446
A_6	0.9991402	0.2355728	0.2477688	0.494160
A_7	0.999954	0.2355729	0.2477689	0.494431
A_8	0.9999253	0.2355735	0.2477696	0.494422
$p_{0.15}$	131.203	-	-	131.203
$p_{0.25}$	180	-	-	180
$p_{0.35}$	204.9558	-	-	204.9558
$p_{0.45}$	226.1095	-	-	226.1095
$p_{0.55}$	256	-	-	256
$p_{0.65}$	280	-	-	280
$p_{0.75}$	290	-	-	290
$p_{0.85}$	323.3433	-	-	323.3433
N		**65**		
n		**20**		

Table 4: Data statistics of dataset -B

	Classical	**M.C.D**	**M.V.E**	**Real values**
$\bar{Y}$	5182.637	5040.39	4997.303	5073.4433
$\bar{X}$	285.125	118.7358	118.7358	174.1988
S_y^2	3369642	4184695	4202338	3918891.667
S_x^2	73132.09	124879.1	124879.1	107630.0967
C_y	0.3541939	0.4058521	0.4102134	0.39008
C_x	0.9484593	2.976206	2.976206	2.30029
$\lambda_{04}=\beta_{2(x)}$	2.864621	3.841065	3.841065	3.51558
$\lambda_{40}=\beta_{2(y)}$	1.813323	1.852405	1.866517	1.84408
λ_{22}	1.858694	2.132329	2.181907	2.05764
β	21.21895	13.35566	13.99921	16.19127

A_1	1	33.50973	33.65101	22.7202
A_2	0.9999608	33.50973	33.65101	**22.72023**
A_3	0.999987	33.50974	33.65102	**22.720249**
A_4	0.9999905	33.50972	33.651	22.720236
A_5	0.9999955	33.50974	33.65102	22.720251
A_6	0.9999587	33.50978	33.65106	22.720266
A_7	0.9999899	33.50975	33.65104	22.720259
A_8	0.9999814	33.50974	33.65102	22.720247
$p_{0.15}$	70.85	-	-	70.85
$p_{0.25}$	86.5	-	-	86.5
$p_{0.35}$	111.95	-	-	111.95
$p_{0.45}$	132.65	-	-	132.65
$p_{0.55}$	169.15	-	-	169.15
$p_{0.65}$	287.1	-	-	287.1
$p_{0.75}$	445.25	-	-	445.25
$p_{0.85}$	586.95	-	-	586.95
N			80	
n			20	

Table 5: Theoretical findings for the PRE for every suggested variance estimator from dataset-A for simple random sampling with regard to Zaman and Bulut (2022).

	Existing (Zaman an Bulut 2022)	**M.C.D**	**M.V.E**
Estimators	M.S.E PRE	M.S.E PRE	M.S.E PRE
S^2_{PR1}	10980375 100	7072365 155.257	7855697 139.775
S^2_{PR2}	10975311 100	7072356 155.186	7855686 139.711
S^2_{PR3}	10980030 100	7072365 155.252	7855687 139.771
S^2_{PR4}	10980104 100	7072355 155.253	7855685 139.772
S^2_{PR5}	10980332 100	7072358 155.257	7855688 139.775
S^2_{PR6}	10971162 100	7072355 155.127	7855685 139.658
S^2_{PR7}	10979882 100	7072355 155.250	7855686 139.769
S^2_{PR8}	10979574 100	7072357 155.246	7855687 139.765

Table 6: Theoretical findings for the PRE for every suggested variance estimator from dataset-A for simple random sampling with regard to robust class of Zaman and Bulut (2022).

Estimators	Existing (Zaman andBulut 2022) M.C.D	Existing (Zaman andBulut 2022) M.V.E	M.C.D	M.V.E
	M.S.E PRE	M.S.E PRE	M.S.E PRE	M.S.E PRE
S^2_{PR1}	12253774 100	13568924 100	7072365 173.262	7855697 172.727
S^2_{PR2}	12248772 100	13563391 100	7072356 173.192	7855686 172.656
S^2_{PR3}	12253429 100	13568543 100	7072365 173.257	7855687 172.722
S^2_{PR4}	12253499 100	13568619 100	7072355 173.259	7855685 172.723
S^2_{PR5}	12253729 100	13568875 100	7072358 173.262	7855688 172.726
S^2_{PR6}	12244428 100	13558585 100	7072355 173.130	7855685 172.595
S^2_{PR7}	12253259 100	13568354 100	7072355 173.255	7855686 172.720
S^2_{PR8}	12252968 100	13568035 100	7072357 173.251	7855687 172.716

Table 7: Theoretical findings for the PRE for every suggested variance estimator from dataset-B for simple random sampling with regard to Zaman and Bulut (2022).

Estimators	Existing (Zaman an Bulut 2022)	M.C.D	M.V.E
	M.S.E PRE	M.S.E PRE	M.S.E PRE
S^2_{PR1}	7.741721e+16 100	2.119067e+13 7337.975	2.143514e+13 7254.285
S^2_{PR2}	7.741114e+16 100	2.119067e+13 7337.400	2.143514e+13 7253.716
S^2_{PR3}	7.74152e+16 100	2.119069e+13 7337.778	2.143515e+13 7254.093
S^2_{PR4}	7.741573e+16 100	2.119066e+13 7337.838	2.143513e+13 7254.150
S^2_{PR5}	7.741651e+16 100	2.119068e+13 7337.905	2.143515e+13 7254.216
S^2_{PR6}	7.741081e+16 100	2.119074e+13 7337.344	2.14352e+13 7253.665
S^2_{PR7}	7.741565e+16 100	2.11907e+13 7337.817	2.143517e+13 7254.129
S^2_{PR8}	7.741433e+16 100	2.119069e+13 7337.695	2.143515e+13 7254.012

Table 8: Theoretical findings for the PRE for every suggested variance estimator from dataset-B for simple random sampling with regard to robust class of Zaman and Bulut (2022).

Estimators	Existing (Zaman an Bulut 2022) M.C.D	Existing (Zaman an Bulut 2022) M.V.E	M.C.D	M.V.E
	M.S.E PRE	M.S.E PRE	M.S.E PRE	M.S.E PRE
S^2_{PR1}	3.106484e+17 100	3.132734e+17 100	2.119067e+13 8003.913	2.143514 e+13 7979.489
S^2_{PR2}	3.106293e+17 100	3.132541e+17 100	2.119067e+13 8003.420	2.143514e+13 7978.998
S^2_{PR3}	3.106336e+17 100	3.132584e+17 100	2.119069e+13 8003.524	2.143515e+13 7979.104
S^2_{PR4}	3.106448e+17 100	3.132696e+17 100	2.119066e+13 8003.824	2.143513e+13 7979.394
S^2_{PR5}	3.106445e+17 100	3.132695e+17 100	2.119068e+13 8003.808	2.143515e+13 7979.386
S^2_{PR6}	3.10642e+17 100	3.132669e+17 100	2.119074e+13 8003.721	2.14352e+13 7979.301
S^2_{PR7}	3.106472e+17 100	3.132721e+17 100	2.11907e+13 8003.870	2.143517e+13 7979.445
S^2_{PR8}	3.106281e+17 100	3.132535e+17 100	2.119069e+13 8003.382	2.143515e+13 7978.979

Simulation study

The simulation methods that were taken into consideration to calculate the M.S.Es of the recommended estimators were outlined and coded in the R program for the simulation investigation.

For numerical comparisons, we make use of the simulation-based study that follows.The models listed below are what we have used:

$Y_i = 5X_i + \varepsilon_i$ Which we produce ε_i and X_i separately and compute Y_i for $i = 1, 2, ..., N$.

(1) X is from $N(1,5)$ and ε is from $N(0,1)$ and independent of X

(2) X is from $N(1,5)$ and ε is from $Exp(1)$ and independent of X

(3) X is from $N(1,5)$ and ε is from $U(0,1)$ and independent of X

The following steps might be used to summarize simulation. X is produced by taking N=65 and generating the distributions above. The ratio of outliers are 5 and we have ensured that the sample selection contains the least amount of outliers possible.

Initially, using SRSWOR (simple random sampling without replacement), the classical estimators provided in sub-sections 3.1.1 and 3.1.2 are generated for each sample size.

Then, for each sample taken, the proposed robust exponential ratio-type estimators, say s^2_{yi} , such as S^2_{PRi} ,obtained using basic random sampling as described in Section 4.

The M.S.E values are acquired for each case with the assistance of (6.1)

$$MSE = \frac{1}{10000}\sum_{i=1}^{10000}\left(s^2_{yi} - S^2_y\right)^2 \tag{6.1}$$

Where S_y^2 is the population variance. The number of iterations is 10000.

Using simple random sampling, sample sizes of $n = 20, 30$ and 40 are determined. For the different sample sizes, Table 9, 10, 11, 12, 13 and 14 displays the M.S.E values for the existing and suggested robust exponential ratio-type estimators for the normal, exponential, and uniform distributions, respectively. These values are computed using (6.1). Tables 9, 10, 11, 12, 13 and 14 indicate that, for all sample sizes in simple random sampling, the robust exponential ratio-type estimator that has been suggested performs better than existing estimators. The theoretical outcomes shown in Tables 5, 6, 7 and 8 are supported by each of these studies. It is noteworthy to emphasize that in the scenario of a data outlier, estimators linked to M.C.D and M.V.E estimations are more effective than those that do not make use of appropriate robust statistics.

Table 9: Simulation findings for the Mean Squared Error (M.S.E) of estimators for different sample sizes in relation to the Normal distribution

n:	**20**			**30**			**40**		
Estimators	**Classical**	**M.C.D**	**M.V.E**	**classical**	**M.C.D**	**M.V.E**	**classical**	**M.C.D**	**M.V.E**
S^2_{PR1}	57556257	3324716	3324716	25559158	1472487	1472487	14364994	825374	825374
S^2_{PR2}	39342093	3250506	3250506	17467682	1439563	1439563	9815624	806886.7	806886.7
S^2_{PR3}	14846617	3025454	3025454	6587630	1339720	1339720	3699434	750826.3	750826.3
S^2_{PR4}	49237594	4091509	4091509	21863582	1812719	1812719	12287136	1016436	1016436
S^2_{PR5}	31664959	3593255	3593255	14057443	1591633	1591633	7898388	892278.4	892278.4
S^2_{PR6}	55337860	3375621	3375621	24573620	1495072	1495072	13810863	838055.8	838055.8
S^2_{PR7}	56658360	3511477	3511477	25160260	1555349	1555349	14140708	871903.1	871903.1
S^2_{PR8}	14808786	3402214	3402214	6570830	1506871	1506871	3689992	844681	844681

Table 10: Simulation findings for the Mean Squared Error (M.S.E) of estimators for different sample sizes in relation to the Exponential distribution

n:	20			30			40		
Estimators	**Classical**	**M.C.D**	**M.V.E**	**Classical**	**M.C.D**	**M.V.E**	**Classical**	**M.C.D**	**M.V.E**
S^2_{PR1}	137303199	9062102	9566744	60996143	4020538	4244608	34294864	2257583	2383502
S^2_{PR2}	58757401	9035618	9538798	26096411	4008777	4232199	14669116	2250973	2376527
S^2_{PR3}	88412031	16218406	17116698	39272168	7198730	7597686	22078185	4043973	4268225
S^2_{PR4}	114091128	10255084	10825502	50682099	4550301	4803592	28494583	2555321	2697669
S^2_{PR5}	125199972	10049530	10608619	55618171	4459019	4707278	31270453	2504018	2643536
S^2_{PR6}	95032410	10236074	10805444	42213748	4541859	4794685	23732367	2550576	2692663
S^2_{PR7}	127186957	10064425	10624335	56501068	4465634	4714257	31766966	2507735	2647459
S^2_{PR8}	88385550	10254500	10824885	39260402	4550042	4803318	22071569	2555175	2697515

Table 11: Simulation findings for the Mean Squared Error (M.S.E) of estimators for different sample sizes in relation to the Uniform distribution

n:	20			30			40		
Estimators	**Classical**	**M.C.D**	**M.V.E**	**Classical**	**M.C.D**	**M.V.E**	**Classical**	**M.C.D**	**M.V.E**
S^2_{PR1}	9103397	930920.1	1073621	4039750	411711.8	474937.8	2268873	230448.6	265902.7
S^2_{PR2}	5469148	971928.6	1119960	2425924	429893.5	495485.3	1361881	240650.9	277433.8
S^2_{PR3}	5641496	1564873	1785232	2502448	692865.6	790563.8	1404883	388258.5	443079.1
S^2_{PR4}	7378824	1092375	1255790	3273892	483300.3	555719.9	1838425	270621.9	311240.3
S^2_{PR5}	7476881	1082126	1244247	3317436	478755.5	550600.8	1862898	268071.3	308367.1
S^2_{PR6}	7275183	1080839	1242797	3227869	478184.6	549957.7	1812560	267750.9	308006.1
S^2_{PR7}	8315854	1059686	1218964	3690006	468804.9	539388.6	2072296	262486.9	302074
S^2_{PR8}	5641012	1055287	1214007	2502233	466854.6	537190.3	1404762	261392.4	300840.1

Table 12: Simulation findings for the Mean Squared Error (M.S.E) of estimators for different sample sizes in relation to the Normal distribution

n:	20			30			40					
	Classical	**M.C.D**	**M.V.E**	**Classical M.C.D**	**M.C.D**	**M.V.E**	**Classical M.C.D**	**M.C.D**	**M.V.E**			
S^2_{PR1}	62429295	62429295	3324716	3324716	27723959	27723959	1472487	1472487	15582135	15582135	825374	825374
S^2_{PR2}	42504168	42504168	3250506	3250506	18872264	18872264	1439563	1439563	10605260	10605260	806886.7	806886.7
S^2_{PR3}	26750019	26750019	3025454	3025454	11874240	11874240	1339720	1339720	6671020	6671020	750826.3	750826.3
S^2_{PR4}	53445980	53445980	4091509	4091509	23733048	23733048	1812719	1812719	13338190	13338190	1016436	1016436

S^2_{PR5}	43768357	43768357	3593255	3593255	19433853	19433853	1591633	1591633	10921000	10921000	892278.4	892278.4
S^2_{PR6}	58365938	58365938	3375621	3375621	25918764	25918764	1495072	1495072	14567130	14567130	838055.8	838055.8
S^2_{PR7}	60793028	60793028	3511477	3511477	26997024	26997024	1555349	1555349	15173401	15173401	871903.1	871903.1
S^2_{PR8}	267706363	267706363	3402214	3402214	11854849	11854849	1506871	1506871	6660120	6660120	844681	844681

Table 13: Simulation findings for the Mean Squared Error (M.S.E) of estimators for different sample sizes in relation to the Exponential distribution

n:	20				30				40			
	Classical	M.C.D M.V.E	M.C.D	M.V.E	Classical	M.C.D M.V.E	M.C.D	M.V.E	Classical	M.C.D M.V.E	M.C.D	M.V.E
S^2_{PR1}	135422692	142284625	9062102	9566744	60160549	63209535	4020538	4244608	33824948	35539574	2257583	2383502
S^2_{PR2}	58213979	60483459	9035618	9538798	25854972	26863233	4008777	4232199	14533353	15100276	2250973	2376527
S^2_{PR3}	88283938	89545960	16218406	17116698	39215253	39775932	7198730	7597686	22046178	22361437	4043973	4268225

S^2_{PR4}	112531999	118250951	10255084	10825502	49989323	52530383	4550301	4803592	28104992	29533947	2555321	2697669
S^2_{PR5}	123755624	129206606	10049530	10608619	54976389	57398403	4459019	4707278	30909535	32271556	2504018	2643536
S^2_{PR6}	92936182	100032682	10236074	10805444	41282343	44435429	4541859	4794685	23208594	24981691	2550576	2692663
S^2_{PR7}	125158935	132338337	10064425	10624335	55599935	58789961	4465634	4714257	31260196	33054126	2507735	2647459
S^2_{PR8}	88257627	89500537	10254500	10824885	39203562	39755750	4550042	4803318	22039604	22350088	2555175	2697515

Table 14: Simulation findings for the Mean Squared Error (M.S.E) of estimators for different sample sizes in relation to the Uniform distribution

n:	20				30				40			
	Classical	**M.C.D M.V.E**	**M.C.D**	**M.V.E**	**Classical**	**M.C.D M.V.E**	**M.C.D**	**M.V.E**	**Classical**	**M.C.D M.V.E**	**M.C.D**	**M.V.E**
S^2_{PR1}	12034814	14389183	930920.1	1073621	5341499	6387032	411711.8	474937.8	3000486	3588118	230448.6	265902.7
S^2_{PR2}	7171872	8597237	971928.6	1119960	3181858	3814687	429893.5	495485.3	1786626	2142216	240650.9	277433.8

S^2_{PR3}	S^2_{PR4}	S^2_{PR5}	S^2_{PR6}	S^2_{PR7}	S^2_{PR8}
8464457	9802820	10486123	8876866	10688027	8463955
10475374	11770193	12736649	10355854	12679458	10474837
1564873	1092375	1082126	1080839	1059686	1055287
1785232	1255790	1244247	1242797	1218964	1214007
3755853	4350214	4653678	3938998	4743348	3755630
4648759	5223816	5653055	4595679	5627654	4648521
692865.6	483300.3	478755.5	478184.6	468804.9	466854.6
790563.8	555719.9	550600.8	549957.7	539388.6	537190.3
2109223	2443289	2613860	2212160	2664263	2109098
2611014	2934248	3175528	2581178	3161249	2610879
388258.5	270621.9	268071.3	267750.9	262486.9	261392.4
443079.1	311240.3	308367.1	308006.1	302074	300840.1

Conclusion

In this study, we propose utilizing M.C.D and M.V.E estimates in simple random sampling to develop a robust exponential ratio-type estimator for population variance. We derive expressions for the mean square error of the suggested estimators, which allow us to compare their performance with that of other estimators considered in this context. Our analysis reveals that the proposed estimators exhibit superior performance under certain conditions. Simulation studies and empirical data confirm that these robust exponential ratio-type estimators consistently achieve lower mean square errors compared to conventional estimators when applied to simple random sampling. Additionally, we plan to extend this work by adapting the robust exponential ratio-type estimators for use in simple two-stage sampling in future research

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